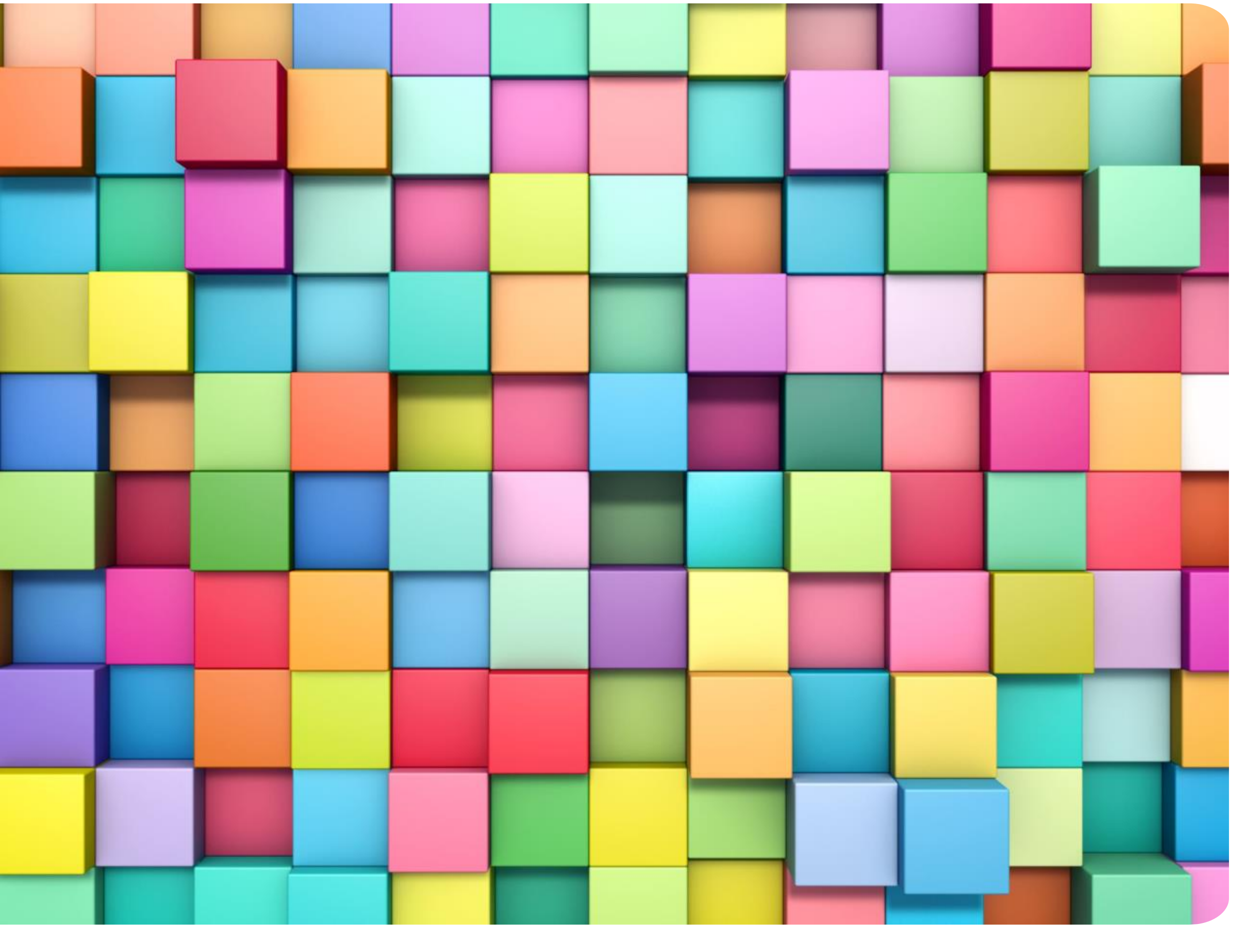

Root locus for sampled data systems

Digital control

Dr. Ahmad Al-Mahasneh

Root locus

- The root locus is one of the most powerful techniques used to analyse the stability of a closed- loop system.
- This technique is also used to design controllers with required time response characteristics. The root locus is a plot of the locus of the roots of the characteristic equation as the gain of the system is varied.
- The rules of the root locus for discrete-time systems are identical to those for continuous systems.
- This is because the roots of an equation $Q(z)$ 0 in the z-plane are the same as the roots of $Q(s)$ 0 in the s-plane.
- Even though the rules are the same, the interpretation of the root locus is quite different in the s-plane and the z-plane. For example, a continuous system is stable if the roots are in the left-hand s-plane.
- A discrete-time system, on the other hand, is stable if the roots are inside the unit circle.

Root locus

Given the closed-loop system transfer function

$$\frac{G(z)}{1 + GH(z)},$$

we can write the characteristic equation as $1 + kF(z) = 0$, and the root locus can then be plotted as k is varied. The rules for constructing the root locus can be summarized as follows:

1. The locus starts on the poles of $F(z)$ and terminate on the zeros of $F(z)$.
2. The root locus is symmetrical about the real axis.
3. The root locus includes all points on the real axis to the left of an odd number of poles and zeros.

Root locus

4. If $F(z)$ has zeros at infinity, the root locus will have asymptotes as $k \to \infty$. The number of asymptotes is equal to the number of poles n_p, minus the number of zeros n_z. The angles of the asymptotes are given by

$$\theta = \frac{180r}{n_p - n_z}, \quad \text{where } r = \pm 1, \pm 3, \pm 5, \ldots .$$

The asymptotes intersect the real axis at σ, where

$$\sigma = \frac{\sum \text{poles of } F(z) - \sum \text{zeros of } F(z)}{n_p - n_z}.$$

5. The breakaway points on the real axis of the root locus are at the roots of

$$\frac{dF(z)}{dz} = 0.$$

6. If a point is on the root locus, the value of k is given by

$$1 + kF(z) = 0 \quad \text{or} \quad k = -\frac{1}{F(z)}.$$

Root locus

Example 8.8

A closed-loop system has the characteristic equation

$$1 + GH(z) = 1 + K\frac{0.368(z + 0.717)}{(z - 1)(z - 0.368)} = 0.$$

Draw the root locus and hence determine the stability of the system.

Solution

Applying the rules:

1. The above equation is in the form $1 + kF(z) = 0$, where

$$F(z) = \frac{0.368(z + 0.717)}{(z - 1)(z - 0.368)}.$$

The system has two poles at $z = 1$ and at $z = 0.368$. There are two zeros, one at $z = -0.717$ and the other at minus infinity. The locus will start at the two poles and terminate at the two zeros.

Root locus

2. The section on the real axis between $z = 0.368$ and $z = 1$ is on the locus. Similarly, the section on the real axis between $z = -\infty$ and $z = -0.717$ is on the locus.
3. Since $n_p - n_z = 1$, there is one asymptote and the angle of this asymptote is

$$\theta = \frac{180r}{n_p - n_z} = \pm 180^\circ \quad \text{for } r = \pm 1.$$

   Note that since the angles of the asymptotes are $\pm 180^\circ$ it is meaningless to find the real axis intersection point of the asymptotes.

Root locus

4. The breakaway points can be found from

$$\frac{dF(z)}{dz} = 0,$$

or

$$0.368(z-1)(z-0.368) - 0.368(z+0.717)(2z-1.368) = 0,$$

which gives

$$z^2 + 1.434z - 1.348 = 0$$

and the roots are at

$$z = -2.08 \text{ and } z = 0.648.$$

5. The value of k at the breakaway points can be calculated from

$$k = -\frac{1}{F(z)}\bigg|_{z=-2.08,0.648}$$

which gives $k = 15$ and $k = 0.196$.

The root locus of the system is shown in Figure 8.3. The locus is a circle starting from the poles, breaking away at $z = 0.648$ on the real axis, and then joining the real axis at $z = -2.08$.

Root locus

$-\infty$

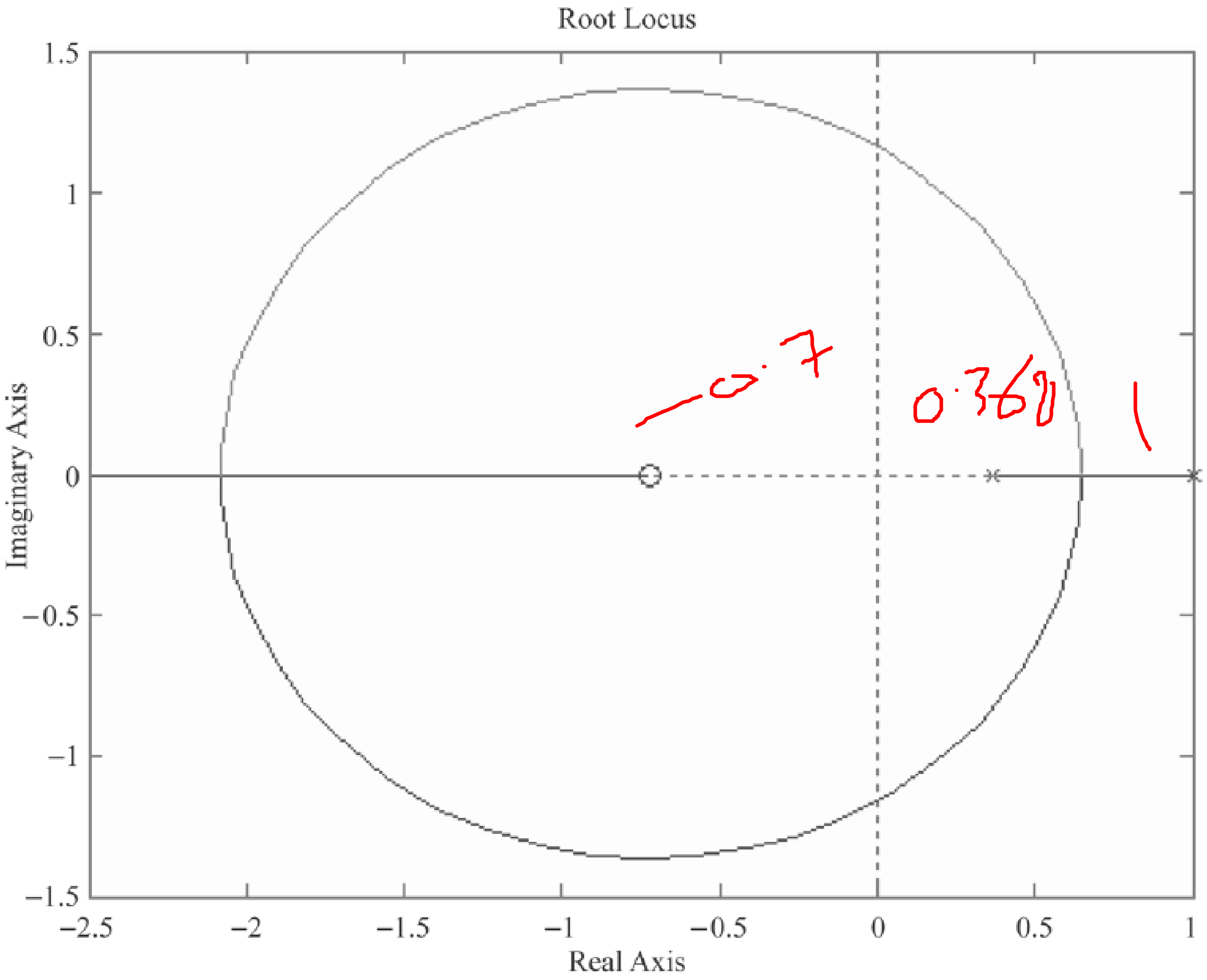

Figure 8.3 Root locus for Example 8.8

Root locus

At this point one part of the locus moves towards the zero at $z = -0.717$ and the other moves towards the zero at $-\infty$.

Figure 8.4 shows the root locus with the unit circle drawn on the same axis. The system will become marginally stable when the locus is on the unit circle. The value of k at these points can be found either from Jury's test or by using the Routh–Hurwitz criterion.

Using Jury's test, the characteristic equation is

$$1 + K\frac{0.368(z + 0.717)}{(z - 1)(z - 0.368)} = 0,$$

or

$$z^2 - z(1.368 - 0.368K) + 0.368 + 0.263K = 0.$$

Applying Jury's test

$$F(1) = 0.631 \quad \text{for } K > 0.$$

Also,

$$|0.263K + 0.368| < 1$$

which gives $K = 2.39$ for marginal stability of the system.

Root locus

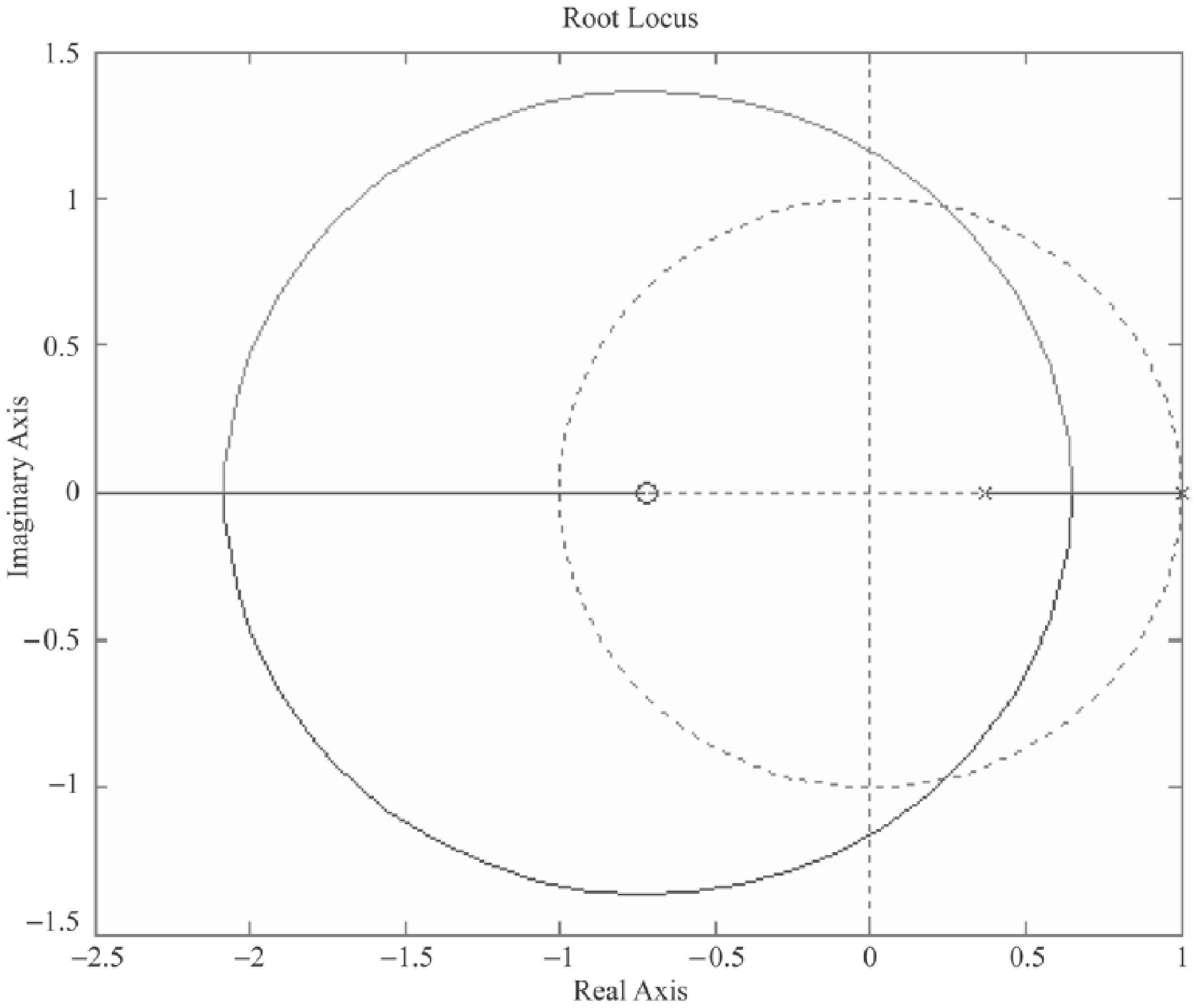

Figure 8.4 Root locus with unit circle

Root locus

Example 8.9

For Example 8.8, calculate the value of k for which the damping factor is $\zeta = 0.7$.

In Figure 8.5 the root locus of the system is redrawn with the lines of constant damping factor and constant natural frequency.

From the figure, the roots when $\zeta = 0.7$ are read as $s_{1,2} = 0.61 \pm j0.25$ (see Figure 8.6). The value of k can now be calculated as

$$k = -\left.\frac{1}{F(z)}\right|_{z=0.61\pm j0.25}$$

which gives $k = 0.324$.

Root locus

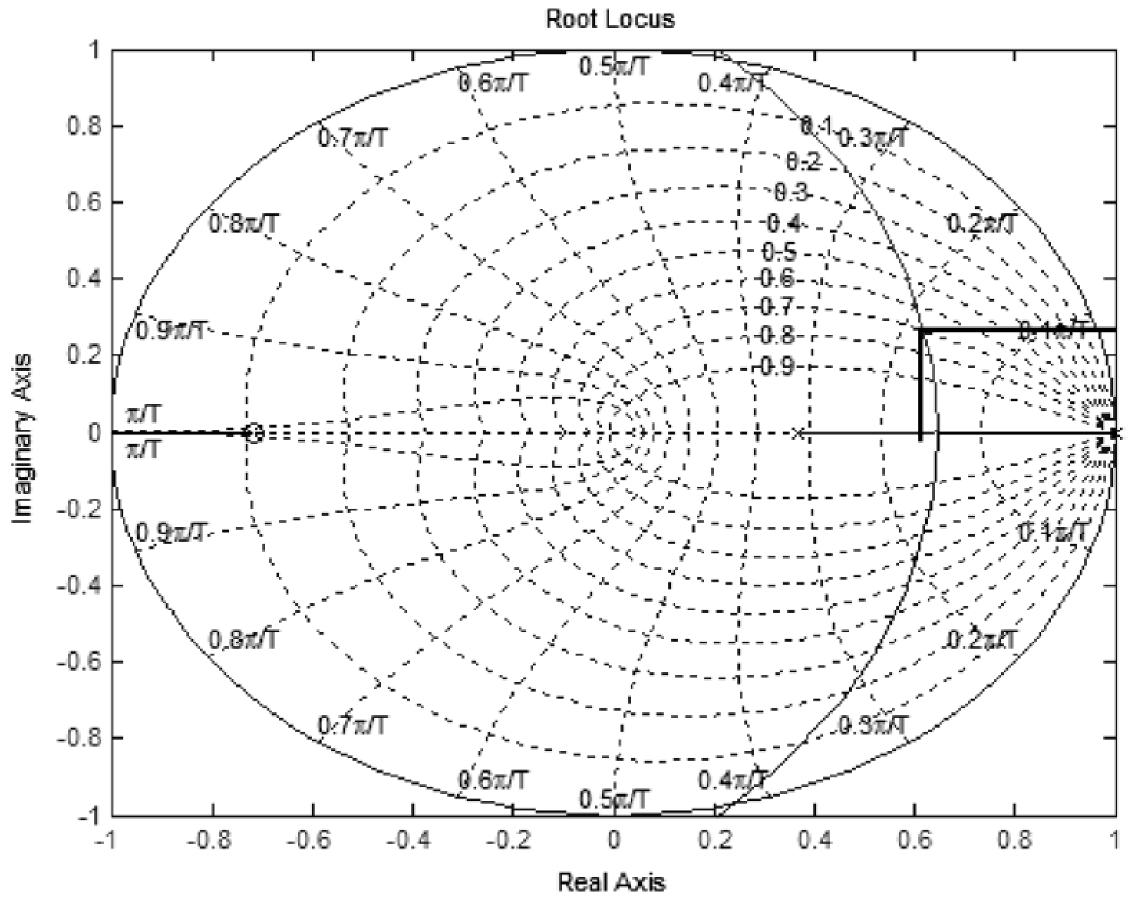

Figure 8.6 Reading the roots when $\zeta = 0.7$

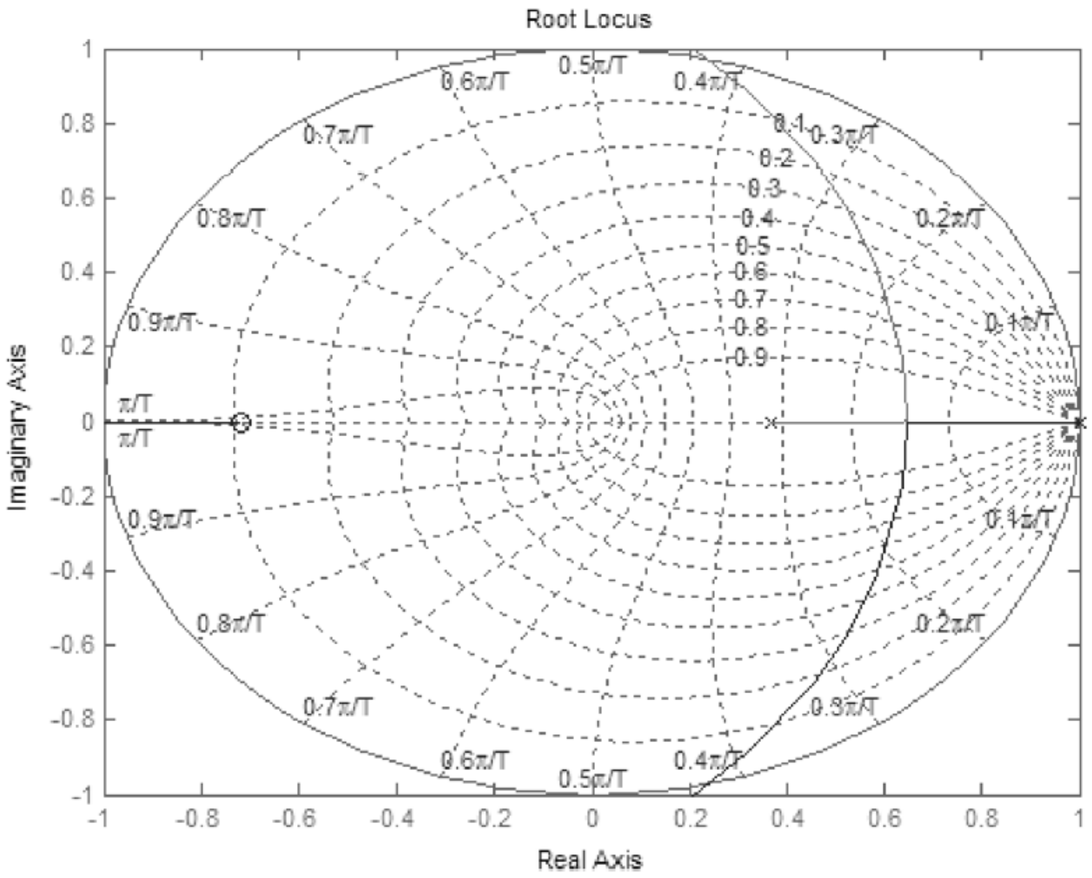

Figure 8.5 Root locus with lines of constant damping factor and natural frequency

Root locus

Example 8.10

A closed-loop system has the characteristic equation

$$1 + GH(z) = 1 + K \frac{(z - 0.2)}{z^2 - 1.5z + 0.5} = 0.$$

Draw the root locus and hence determine the stability of the system. What will be the value of K for a damping factor $\zeta > 0.6$ and a natural frequency of $\omega_n > 0.6$ rad/s?

Root locus

Solution

The above equation is in the form $1 + kF(z) = 0$, where

$$F(z) = \frac{z - 0.2}{z^2 - 1.5z + 0.5}.$$

The system has two poles at $z = 1$ and at $z = 0.5$. There are two zeros, one at $z = -0.2$ and the other at infinity. The locus will start from the two poles and terminate at the two zeros.

1. The section on the real axis between $z = 0.5$ and $z = 1$ is on the locus. Similarly, the section on the real axis between $z = -\infty$ and $z = 0.2$ is on the locus.
2. Since $n_p - n_z = 1$, there is one asymptote and the angle of this asymptote is

$$\theta = \frac{180r}{n_p - n_z} = \pm 180^\circ \quad \text{for } r = \pm 1$$

Note that since the angle of the asymptotes are $\pm 180^\circ$ it meaningless to find the real axis intersection point of the asymptotes.

Root locus

3. The breakaway points can be found from

$$\frac{dF(z)}{dz} = 0$$

or

$$(z^2 - 1.5z + 0.5) - (z - 0.2)(2z - 1.5) = 0,$$

which gives

$$z^2 - 0.4z - 0.2 = 0$$

and the roots are at

$$z = -0.290 \quad \text{and} \quad z = 0.689.$$

4. The value of k at the breakaway points can be calculated from

$$k = -\frac{1}{F(z)}\bigg|_{z=-0.290,0.689}$$

which gives $k = 0.12$ and $k = 2.08$. The root locus of the system is shown in Figure 8.7. It is clear from this plot that the system is always stable since all poles are inside the unit circle for all values of k.

Lines of constant damping factor and constant angular frequency are plotted on the same axis in Figure 8.8.

Root locus

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Lines of constant damping factor and constant angular frequency are plotted on the same axis in Figure 8.8.

Root locus

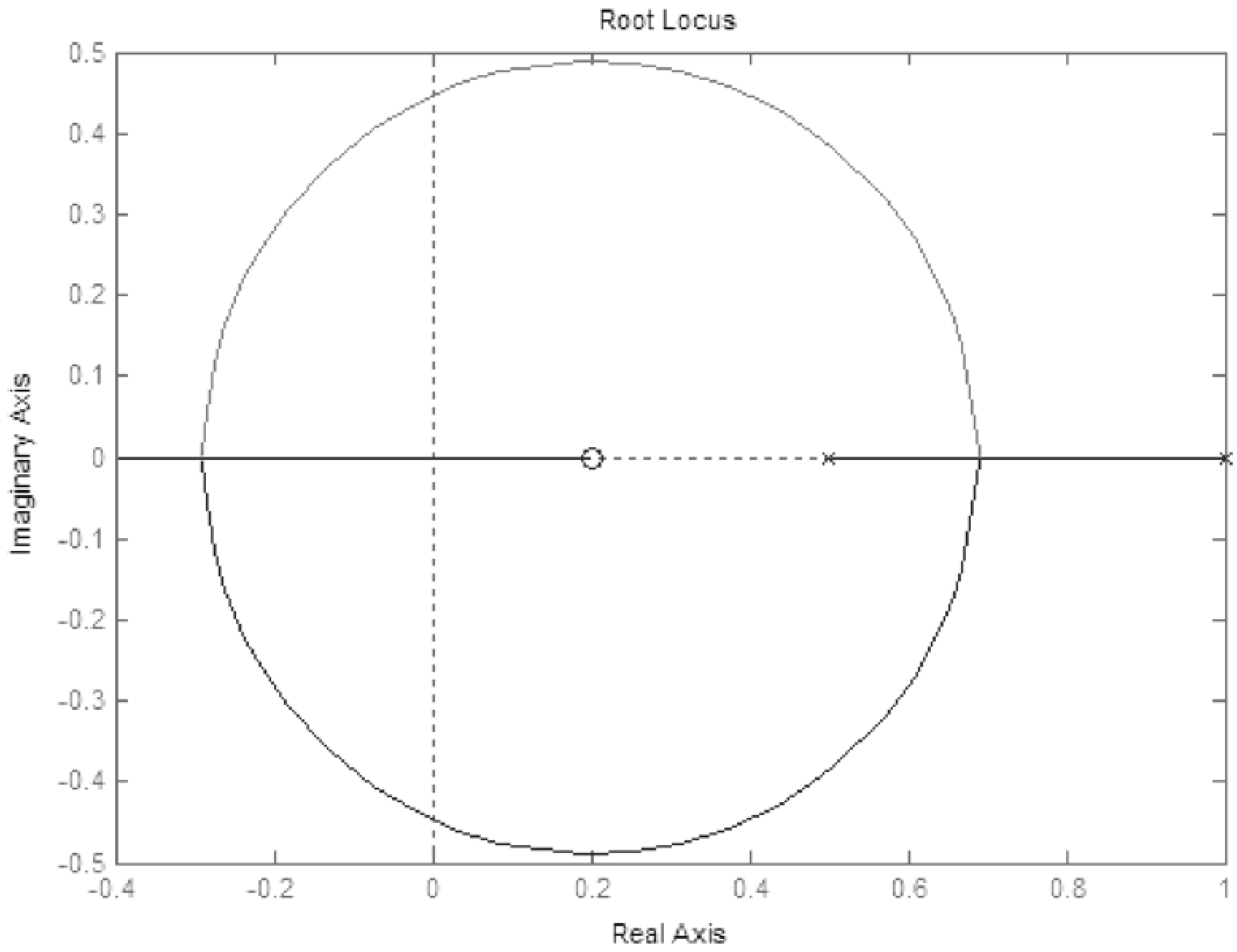

Figure 8.7 Root locus for Example 8.10

Root locus

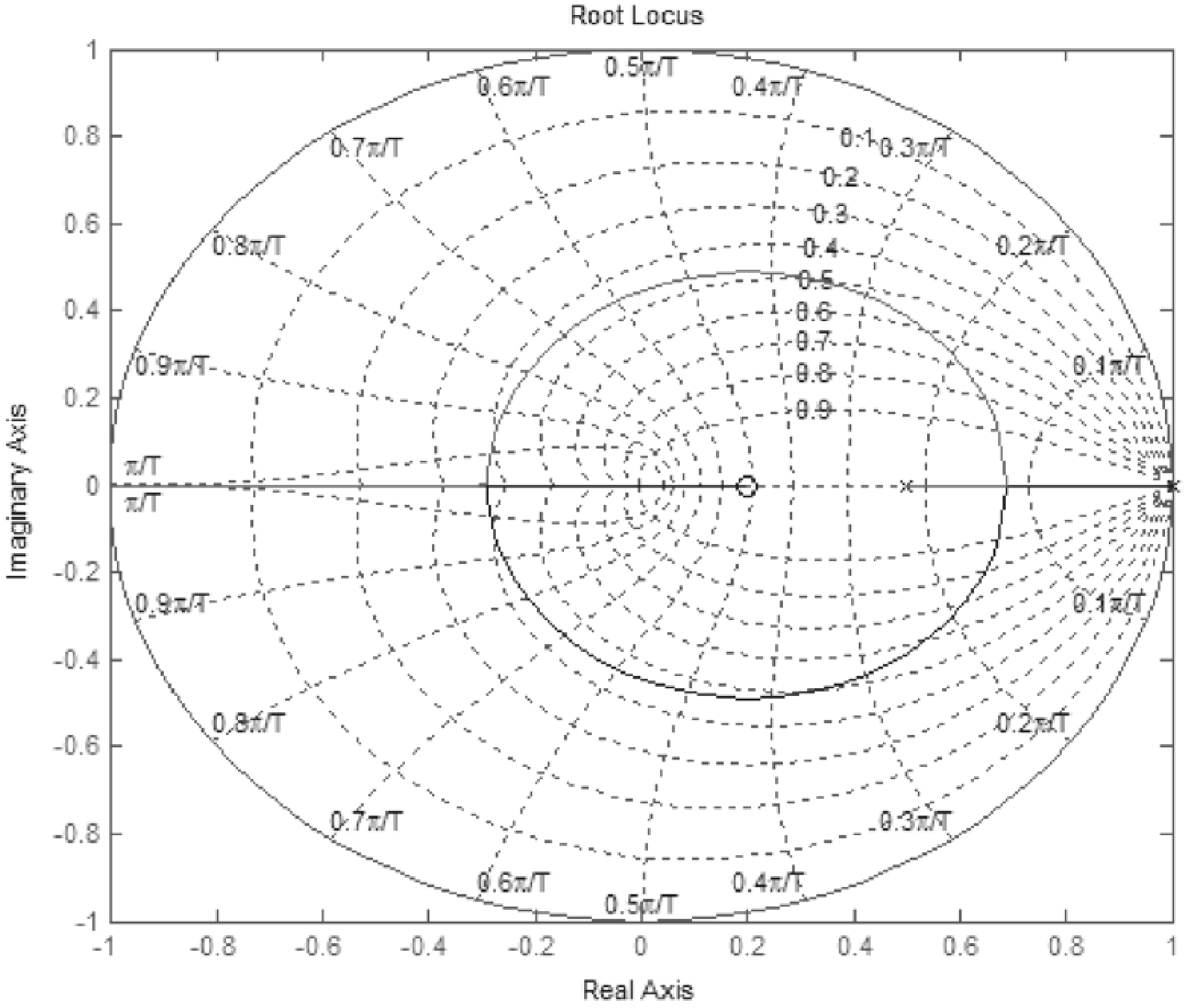

Figure 8.8 Root locus with lines of constant damping factor and natural frequency

Root locus

Assuming that $T = 1$ s, $\omega_n > 0.6$ if the roots are on the left-hand side of the constant angular frequency line $\omega_n = 0.2\pi/T$. The damping factor will be greater than 0.6 if the roots are below the constant damping ratio line $\zeta = 0.6$. A point satisfying these properties has been chosen and shown in Figure 8.9. The roots at this point are given as $s_{1,2} = 0.55 \pm j0.32$. The value of k can now be calculated as

$$k = -\frac{1}{F(z)}\bigg|_{z=0.55\pm j0.32}$$

which gives $k = 0.377$.

Root locus

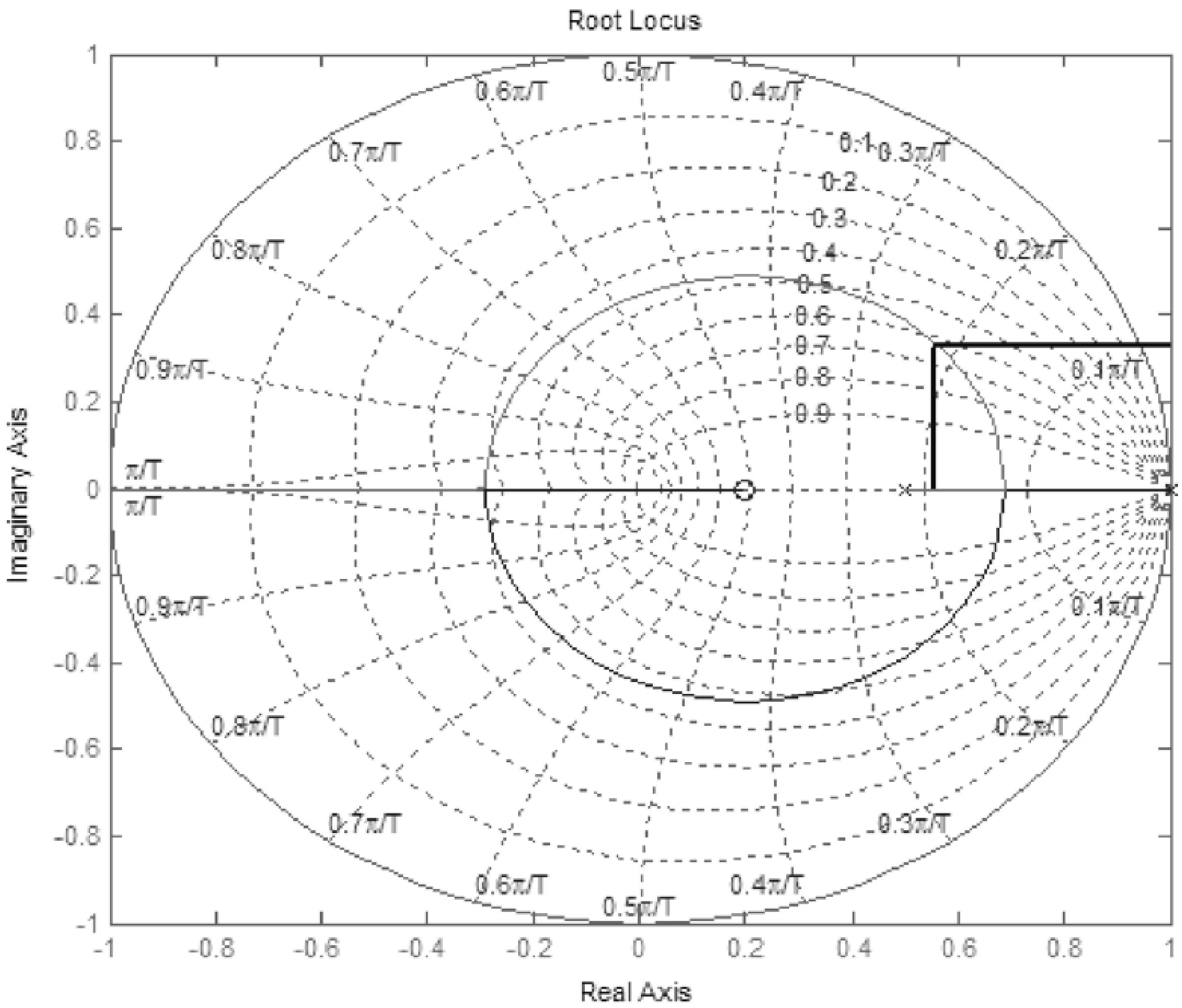

Figure 8.9 Point for $\zeta > 0.6$ and $\omega_n > 0.6$